1 - 10 Spectrum
Are the following matrices symmetric, skew-symmetric, or orthogonal? Find the spectrum of each, thereby illustrating theorem 1, p. 335, and theorem 5, p. 337.

1. $\begin{pmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{pmatrix}$

```
ClearAll["Global`*"]
```

```
aA = (0.8   0.6
      -0.6  0.8)
```

```
{{0.8, 0.6}, {-0.6, 0.8}}
```

```
e1 = Transpose[aA] // MatrixForm
```

$\begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix}$

```
e2 = Inverse[aA] // MatrixForm
```

$\begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix}$

The transpose gives the inverse; therefore the matrix is orthogonal. It is neither skew-symmetric nor symmetric. The text agrees with the stated classification.

```
e3 = {vals, vecs} = Eigensystem[aA]
```

```
{{0.8 + 0.6 i, 0.8 - 0.6 i},
 {{0. - 0.707107 i, 0.707107 + 0. i}, {0. + 0.707107 i, 0.707107 + 0. i}}}
```

```
e4 = e3[[1]]
```

```
{0.8 + 0.6 i, 0.8 - 0.6 i}
```

Above: the spectrum of aA. The answer agrees with the text. The problem description did not ask for the eigenvectors, nevertheless, I believe the text answer shows eigenvectors. The eigenvectors of Mathematica prove sat below; however, what I take as a text answer eigenvector fails.

```
e5 = aA.vecs[[1]] == vals[[1]] vecs[[1]]
```

```
True
```

```
e6 = aA.vecs[[2]] == vals[[2]] vecs[[2]]
```

```
True
```

```
{{1} + {i}}†
{{1 - i}}

{{1} + {i}}ᵀ
{{1 + i}}

e7 = aA.vecs[[1]] == vals[[1]] {1 - i}
False

e7 = aA.vecs[[1]] == vals[[1]] {1 + i}
False
```

Above: Both flavors of transpose were tried in the test of the text answer's eigenvector, but neither passed.

3. $\begin{pmatrix} 2 & 8 \\ -8 & 2 \end{pmatrix}$

```
ClearAll["Global`*"]
```

bB = $\begin{pmatrix} 2 & 8 \\ -8 & 2 \end{pmatrix}$

```
{{2, 8}, {-8, 2}}
```

```
e1 = Transpose[bB] // MatrixForm
```

$\begin{pmatrix} 2 & -8 \\ 8 & 2 \end{pmatrix}$

```
e2 = Inverse[bB]
```

$\{\{\frac{1}{34}, -\frac{2}{17}\}, \{\frac{2}{17}, \frac{1}{34}\}\}$

The matrix is neither symmetric, skew-symmetric, nor orthogonal. The text only states not skew-symmetric.

```
{vals, vecs} = Eigensystem[bB]
{{2 + 8 i, 2 - 8 i}, {{-i, 1}, {i, 1}}}

e4 = e3[[1]]

{2 + 8 i, 2 - 8 i}
```

Above: the spectrum of **bB,** in disagreement with the text. The text answer shows the spectrum as $2 \pm 0.8i$. The eigenvalues found by Mathematica seem to check out okay below.

```
e5 = bB.vecs[[1]] == vals[[1]] vecs[[1]]
True
```

```
e6 = bB.vecs[[2]] == vals[[2]] vecs[[2]]
True
```

5. $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{pmatrix}$

cC = $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{pmatrix}$

```
{{6, 0, 0}, {0, 2, -2}, {0, -2, 5}}
```

```
e1 = Transpose[cC] // MatrixForm
```

$\begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{pmatrix}$

```
e2 = Inverse[cC]
```

$\left\{\left\{\frac{1}{6}, 0, 0\right\}, \left\{0, \frac{5}{6}, \frac{1}{3}\right\}, \left\{0, \frac{1}{3}, \frac{1}{3}\right\}\right\}$

The matrix is symmetric. It is not skew-symmetric nor orthogonal. This finding is in agreement with the text.

```
{vals, vecs} = Eigensystem[cC]
{{6, 6, 1}, {{0, -1, 2}, {1, 0, 0}, {0, 2, 1}}}
```

```
e4 = e3[[1]]
```

```
{6, 6, 1}
```

Above: the spectrum of **cC**. This answer agrees with the text. Two of the eigenvectors agree with the text. All three of the eigenvectors of Mathematica check out below. The eigenvector which the text answer proposes in place of vals[[1]] seems to fail a test.

```
e5 = cC.vecs[[1]] == vals[[1]] vecs[[1]]
True
```

```
e6 = cC.vecs[[2]] == vals[[2]] vecs[[2]]
True
```

```
e7 = cC.vecs[[3]] == vals[[3]] vecs[[3]]
True
```

```
{{0}, {1}, {-2}}†
{{0, 1, -2}}

{{0}, {1}, {-2}}ᵀ
{{0, 1, -2}}

e8 = cC.vecs[[2]] == vals[[2]] {0, 1, -2}
False
```

In this case both types of transpose yield the same vector, so the test is not affected.

7. $\begin{pmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{pmatrix}$

```
ClearAll["Global`*"]
```

$aA = \begin{pmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{pmatrix}$

```
{{0, 9, -12}, {-9, 0, 20}, {12, -20, 0}}

e1 = Transpose[aA] // MatrixForm
```

$\begin{pmatrix} 0 & -9 & 12 \\ 9 & 0 & -20 \\ -12 & 20 & 0 \end{pmatrix}$

```
e2 = Inverse[aA] // MatrixForm
```

Inverse::sing: Matrix{{0, 9, −12}, {−9, 0, 20}, {12, −20, 0}} is singular ≫

```
Inverse[{{0, 9, -12}, {-9, 0, 20}, {12, -20, 0}}]
```

The matrix is skew-symmetric. It is not orthogonal. The text agrees with the stated classification.

```
e3 = Eigensystem[aA]
{{25 i, -25 i, 0},
 {{-45 + 75 i, -27 - 125 i, 136}, {-45 - 75 i, -27 + 125 i, 136}, {20, 12, 9}}}

e4 = e3[[1]]
{25 i, -25 i, 0}
```

Above: the spectrum of **aA.** The spectrum is in agreement with the text. The text answer does not show eigenvectors for this problem.

9. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

```
ClearAll["Global`*"]
```

$$aA = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

```
{{0, 0, 1}, {0, 1, 0}, {-1, 0, 0}}
```

```
e1 = Transpose[aA] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

```
e2 = Inverse[aA] // MatrixForm
```

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

The matrix is orthogonal. It is neither symmetric nor skew-symmetric. The text agrees with the stated classification.

```
{vals, vecs} = Eigensystem[aA]
{{i, -i, 1}, {{-i, 0, 1}, {i, 0, 1}, {0, 1, 0}}}
```

```
vals
```

```
{i, -i, 1}
```

The spectrum is shown above, and it is in agreement with that shown in the text answer. Two eigenvectors which are shown in the text do not agree with those found by Mathematica. The Mathematica eigenvectors check out. The two from the text in contention do not.

```
e5 = aA.vecs[[1]] == vals[[1]] vecs[[1]]
True
```

```
e6 = aA.vecs[[2]] == vals[[2]] vecs[[2]]
True
```

```
e7 = aA.vecs[[3]] == vals[[3]] vecs[[3]]
True
```

```
{{1}, {0}, {i}}†
{{1, 0, -i}}
```

```
{{1}, {0}, {i}}ᵀ
{{1, 0, i}}
```

```
e8 = aA.vecs[[1]] == vals[[1]] {1, 0, -i}
False
```

```
e9 = aA.vecs[[1]] == vals[[1]] {1, 0, i}
False
```

```
{{1}, {0}, {-i}}†
{{1, 0, i}}
```

```
{{1}, {0}, {-i}}ᵀ
{{1, 0, -i}}
```

```
e10 = aA.vecs[[2]] == vals[[2]] {1, 0, i}
False
```

```
e11 = aA.vecs[[2]] == vals[[2]] {1, 0, -i}
False
```

Above: In this case the different types of transpose yielded different vectors, in two instances. However, the test of the text answer's eigenvalues is not affected.

17. Skew-symmetric matrix. Show that the inverse of a skew-symmetric matrix is skew-symmetric.

I get my example matrix from Wikipedia. When the problem says "show", it usually refers to a proof. However, I am only going to show in the sense of showing an example.

$$A = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & -4 \\ 1 & 4 & 0 \end{pmatrix}$$

```
{{0, 2, -1}, {-2, 0, -4}, {1, 4, 0}}
```

Testing to see if it meets the definition of skew-symmetric.

```
Simplify[-A == Aᵀ]
True
```

```
Inverse[A]
```

Inverse::sing: Matrix{{0, 2, −1}, {−2, 0, −4}, {1, 4, 0}} is singular≫

```
Inverse[{{0, 2, -1}, {-2, 0, -4}, {1, 4, 0}}]
```

Oops, singular matrix, no inverse. I'd better find another one. Here it is

$$A = \begin{pmatrix} 0 & -5 & 4 \\ 5 & 0 & -1 \\ -4 & 1 & 0 \end{pmatrix}$$

```
{{0, -5, 4}, {5, 0, -1}, {-4, 1, 0}}
```

```
Simplify[-A == Aᵀ]
```

```
True
```

```
Inverse[A]
```

Inverse::sing: Matrix{{0, -5, 4}, {5, 0, -1}, {-4, 1, 0}} is singular≫

```
Inverse[{{0, -5, 4}, {5, 0, -1}, {-4, 1, 0}}]
```

Oops, a second singular matrix. I am starting to think that these might be the norm. However, at *https://www.sciencedirect.com/science/article/pii/0024379594001995*, I found an article specifically about non-singular skew-symmetric matrices. Here's one:

$$A = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

```
{{0, 1, -1, 1}, {-1, 0, 1, 1}, {1, -1, 0, 1}, {-1, -1, -1, 0}}
```

```
Simplify[-A == Aᵀ]
```

```
True
```

```
B = Inverse[A]
```

$$\left\{\left\{0, -\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right\}, \left\{\frac{1}{3}, 0, -\frac{1}{3}, -\frac{1}{3}\right\}, \left\{-\frac{1}{3}, \frac{1}{3}, 0, -\frac{1}{3}\right\}, \left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right\}\right\}$$

```
Simplify[-B == Bᵀ]
```

```
True
```

The above cell demonstrates the proposition of the problem, for this example.